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Some results for the entry of a blunt wedge into water

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This paper concerns the motion of water, initially at rest in a half space with horizontal free surface, due to the entry of an infinite wedge moving vertically downwards with constant velocity. The effects of gravity, viscosity and surface tension are neglected. This problem was formulated by Wagner in 1932 and has been the subject of many papers since then, but it seems that there has been no proof, until now, of the existence of a solution.

A recent existence theory by the authors includes an explicit solution, for the limiting case of wedge angle π, that is surprisingly simple in greatly transformed variables. Iteration yields asymptotic formulae for the motion due to wedges of angle slightly less than π; these asymptotic results are presented here.

1. Introduction

Consider the following idealized situation. Initially, at time $T = 0$, liquid at rest occupies the half space $\{(x^*, y^*)|y^* < 0\}$. (The * distinguishes physical dimensional variables from the reduced non-dimensional variables to be introduced presently.) An infinite wedge, of vertex angle $2\pi\alpha$ $(0 < \alpha < \frac{1}{2})$, moves downwards with constant speed $V > 0$ for all time; its vertex meets the origin $(0, 0)$ at time $T = 0$. We wish to describe the motion of the liquid for times $T > 0$.

Let (u^*, v^*) be the fluid velocity and let (a^*, b^*) be material coordinates, relative to the spatial coordinates (x^*, y^*). The particle labelled (a^*, b^*) has position $(x^*, y^*) = (a^*, b^*)$ at time $T = 0$, and

$$(u^*, v^*) = \left(\frac{\partial x^*}{\partial T}, \frac{\partial y^*}{\partial T}\right)\Bigg|_{(a^*,b^*)\,\text{fixed}}$$

under the fundamental map $(a^*, b^*, T) \mapsto (x^*, y^*)$. Whether, in any given statement, u^* and v^* are to be regarded as functions of (a^*, b^*, T) or (x^*, y^*, T) or (x, y) or ... will be implied by the context throughout the paper.

Let Ω^* denote the open set in the (x^*, y^*)-plane shown in figure 1*a* and bounded by the wedge face AB, the free boundary BC and the vertical line DA below the vertex A of the wedge. The curve BC is to have the complex representation

$$z^* = Z^*(a^*, T), \quad 0 < a^* < \infty, \quad T \geqslant 0, \tag{1.1}$$

where $z^* = x^* + \mathrm{i}y^*$ and a^* is the material coordinate introduced above. We seek

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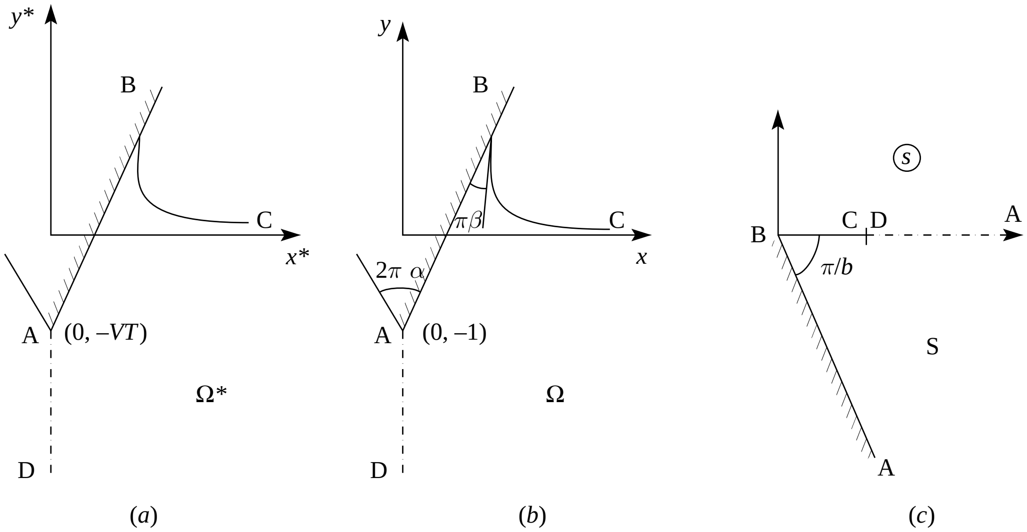

Figure 1. Notation.

a complex velocity field $u^* - \mathrm{i}v^*$ and a free-boundary function Z^* satisfying six conditions. (Explanations will follow this list.)

(I) $u^* - \mathrm{i}v^*$ is to be holomorphic in Ω^* (as a function of z^*, for each fixed $T > 0$) and continuous on the closure $\overline{\Omega^*}$.

(II) On the wedge face AB, the normal velocity of the fluid is that of the wedge:

$$u^* \cos \pi\alpha - v^* \sin \pi\alpha = V \sin \pi\alpha \quad \text{on AB.} \tag{1.2}$$

(III) Symmetry about the y^*-axis implies that

$$u^* = 0 \quad \text{on AD.} \tag{1.3}$$

(IV) The far velocity field is to be no larger than that of a dipole:

$$u^* - \mathrm{i}v^* = O(|z^*|^{-2}) \quad \text{as } |z^*| \to \infty. \tag{1.4}$$

(V) The free boundary BC is to be a material curve:

$$\frac{\partial Z^*}{\partial T}(a^*, T) = (u^* + \mathrm{i}v^*)|_{z^* = Z^*(a^*,T)}, \quad 0 < a^* < \infty, \quad T > 0. \tag{1.5}$$

(VI) The pressure on BC is constant:

$$\mathrm{Re}\, \frac{\partial \overline{Z^*}}{\partial a^*} \frac{\partial^2 Z^*}{\partial T^2} = 0, \quad 0 < a^* < \infty, \quad T > 0, \tag{1.6}$$

where the arguments a^* and T are implied, Re and Im denote the real part and imaginary part, respectively, and the bar denotes complex conjugation ($\overline{z^*} := x^* - \mathrm{i}y^*$).

Condition (I) states that in Ω^* the velocity field (u^*, v^*) is divergence free (the fluid being incompressible) and irrotational (the fluid being inviscid). Conditions (II)–(IV) require no explanation. Equation (1.5) ensures that a^* is indeed a material coordinate on BC. To derive (1.6) we introduce the fluid pressure $p^*(x^*, y^*, T)$ and the constant fluid density $\rho > 0$. Invoking once more the fundamental map $(a^*, b^*, T) \mapsto (x^*, y^*)$, we write the momentum equation of the fluid as

$$\left(\frac{\partial^2 x^*}{\partial T^2}, \frac{\partial^2 y^*}{\partial T^2}\right)\bigg|_{(a^*,b^*)\text{ fixed}} = -\frac{1}{\rho}\left(\frac{\partial p^*}{\partial x^*}, \frac{\partial p^*}{\partial y^*}\right). \tag{1.7}$$

Here gravity g has been neglected; the conventional excuse is that gT/V is assumed to be small. We take the tangential component on BC of the momentum equation by forming the scalar product of (1.7) and the tangential vector $(\partial X^*/\partial a^*, \partial Y^*/\partial a^*)$, where $X^* + \mathrm{i}Y^* = Z^*$; then (1.6) follows.

Wagner (1932) recognized, among much else, that time can be removed from the problem by the similarity transformation

$$z^* = VTz, \quad z = x + \mathrm{i}y, \quad T > 0, \tag{1.8 a}$$

$$(u^* - \mathrm{i}v^*)(z^*, T) = V(u - \mathrm{i}v)(z), \tag{1.8 b}$$

$$Z^*(a^*, T) = VTZ(a), \quad \text{where } a = a^*/(VT). \tag{1.8 c}$$

The new variables z, $u - \mathrm{i}v$, Z and a will be called *reduced* variables. The reduced domain (the same for all $T > 0$) is $\Omega = \{z | VTz \in \Omega^*\}$.

We seek complex-valued functions $Z \in C^1[0, \infty) \cap C^2(0, \infty)$ and $u - \mathrm{i}v \in C(\overline{\Omega})$, with $u - \mathrm{i}v$ holomorphic in Ω, such that the reduced versions of (II)–(VI) hold. Conditions (II)–(IV) involve no time derivative and the transformed versions of these are obvious.

The material-curve condition (V) now extends to $a = 0$ by continuity; it becomes

$$Z(a) - aZ'(a) = (u + \mathrm{i}v)|_{z=Z(a)} \quad (0 \leqslant a < \infty), \tag{1.9}$$

where $(\cdot)'$ denotes diferentiation. The constant-pressure condition (VI) becomes

$$\mathrm{Re}\, \overline{Z'} Z'' = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}a}|Z'|^2 = 0 \quad (0 < a < \infty), \tag{1.10}$$

where the argument a is implied.

Careful use of (1.4), (1.9) and (1.10) yields

$$Z(a) = a + O(a^{-2}) \quad \text{and} \quad Z'(a) = 1 + O(a^{-3}) \quad \text{as } a \to \infty; \tag{1.11}$$

$$|Z'(a)| = 1, \quad 0 \leqslant a < \infty. \tag{1.12}$$

This last means that $|\partial Z^*(a^*, T)/\partial a^*| = 1$ and hence that *on the free boundary, arc length between material points is conserved*. Moreover, (1.12) allows us to write the unit tangent to BC at the point a as

$$Z'(a) = \mathrm{e}^{\mathrm{i}\vartheta(a)}, \quad 0 \leqslant a < \infty. \tag{1.13}$$

We shall pursue the function ϑ as our principal unknown; it satisfies $-\pi < \vartheta(a) < 0$ if BC is as in figure 1.

The condition $Z \in C^1[0, \infty)$ implies existence of a *contact angle* $\pi\beta$, which is shown in figure 1(*b*) and is defined to be the angle between the wedge face and the tangent to BC at the *contact point* B. Accepting the arguments of Garabedian (1965), Dobrovol'skaya (1969) and Mackie (1969) that $0 < \pi\beta < \frac{1}{4}\pi$ for all wedge angles $2\pi\alpha$ ($0 < \alpha < \frac{1}{2}$), we shall prescribe the parameter $\beta \in (0, \frac{1}{4})$ and calculate the corresponding α *a posteriori*. Consequently, as $\alpha \downarrow 0$, the contact angle $\pi\beta$ does not tend to $\frac{1}{2}\pi$. This may seem strange, but there is strong evidence that, for small $\alpha > 0$, there is a region near B, of width proportional to α, in which disturbances are not small; in other words, that there is a boundary-layer phenomenon near the contact point as $\alpha \downarrow 0$.

2. An integral equation

It seems desirable to give some indication of the route to the formulae and graphs in § 4 and § 5, even though only a sketch is possible here. We begin with some key equations from the paper on the existence theory (McLeod & Fraenkel 1998); that long paper will be denoted by LP.

Preliminary observations in LP, beginning with a formulation close to that used by Dobrovol'skaya (1969) for numerical calculation, prompt the following construction. Recall that (unlike Dobrovol'skaya) we prescribe the contact angle $\pi\beta$ with $0 < \beta < \frac{1}{4}$.

(i) Consider the function $z_h = z_h(\,\cdot\,;\beta)$ such that the transformation $z = z_h(s)$ maps the sector

$$S = \left\{ s \in \mathbb{C} \,\middle|\, |s| > 0, -\frac{\pi}{b} < \arg s < 0 \right\}, \quad \text{where } b = \frac{2}{1-4\beta}, \tag{2.1}$$

conformally onto the set Ω in the z-plane in the manner shown by figure 1*c*.

(*a*) To write the function z_h, we pretend that we know not only β, but also a function $h = h(\cdot;\beta)$ defined by the equation

$$\vartheta(a(s)) = -\pi b \int_s^1 h(\sigma)\,\mathrm{d}\sigma, \quad 0 \leqslant s < 1, \tag{2.2}$$

where ϑ is as in (1.13). This implies that there is to be an increasing function $s \mapsto a$ for s real and $0 \leqslant s < 1$; then $h(s)$ *is a weighted curvature of the free boundary* BC. We shall say that h is *admissible* if $h \in C[0,1]$, if the function with values $(1-s)^{1/2}h'(s)$ is also in $C[0,1]$ and if $h(1) = 0$. Limiting values are taken at $s = 0$ and $s = 1$ throughout the paper. The geometry at the contact point B in the z-plane shows that (since $\vartheta(0) < 0$)

$$\left(\tfrac{1}{2}\pi - \pi\alpha\right) + \pi\beta - \vartheta(0) = \pi,$$

whence

$$\tfrac{1}{2} + \alpha - \beta = -\frac{\vartheta(0)}{\pi} = b\int_0^1 h(\sigma)\,\mathrm{d}\sigma. \tag{2.3}$$

This equation determines α in terms of β and $h(\,\cdot\,;\beta)$.

(*b*) The complex logarithmic potential

$$(\mathcal{M}h)(s) = b\int_0^1 \log\left(\frac{\sigma^b - s^b}{\sigma^b}\right) h(\sigma)\,\mathrm{d}\sigma, \quad s \in \overline{S}, \tag{2.4}$$

in which $\arg(\sigma^b - s^b) \in [0,\pi]$ and $\arg\sigma^b = 0$, is an essential ingredient of the mapping function z_h. Note that $(\mathcal{M}h)(0) = 0$ at the image $s = 0$ of the contact point B.

(*c*) The formula for z_h is

$$z_h(s_0) = -\mathrm{i} + \mathrm{i}P\int_{s_0}^{\infty} G(s)\,\mathrm{d}s, \quad s_0 \in \overline{S}\setminus\{1\}, \tag{2.5 a}$$

where

$$G(s) = b\mathrm{e}^{\mathrm{i}\pi(\beta-\alpha)} s^{b\beta-1}(1-s^b)^{-3/2}\exp\{(\mathcal{M}h)(s)\}, \tag{2.5 b}$$

$\arg(1-s^b) \in [0,\pi]$ and P is a real positive constant to be evaluated presently.

(ii) The map z_h is accompanied by the complex velocity field

$$(u - \mathrm{i}v)(s_0) = \mathrm{i}Q\int_{s_0}^{1} H(s)\,\mathrm{d}s, \quad s_0 \in \overline{S}, \tag{2.6 a}$$

where

$$H(s) = b\mathrm{e}^{\mathrm{i}\pi(3/2-\beta+\alpha)}s^{b/2-b\beta-1}\exp\{-(\mathcal{M}h)(s)\}, \qquad (2.6\,b)$$

and Q is a second positive constant. Condition (1.9), at the contact point $a = 0$, demands that $z_{\mathrm{B}} = (u + \mathrm{i}v)_{\mathrm{B}}$; it follows, after a contour integration of H, that

$$P = \frac{\displaystyle\int_0^\infty |H(r\mathrm{e}^{-\mathrm{i}\pi/b})|\,\mathrm{d}r}{\left\{\displaystyle\int_0^\infty |G(r\mathrm{e}^{-\mathrm{i}\pi/b})|\,\mathrm{d}r\right\}\left\{\displaystyle\int_1^\infty |H(s)|\,\mathrm{d}s\right\}}, \quad Q = \frac{1}{\displaystyle\int_1^\infty |H(s)|\,\mathrm{d}s}, \qquad (2.7)$$

where the integrals are along the boundary of S.

(iii) Equations (2.5)–(2.7) ensure that conditions (I)–(IV) and (VI) hold for all admissible functions h; this is proved in LP. It remains to satisfy the material-curve condition (V) or (1.9) for $a > 0$; this condition will hold if h satisfies the integral equation

$$h(s) = K(h)\frac{\exp\{-(Mh)(s)\}}{b\beta s^{-b\beta}\displaystyle\int_0^s \sigma^{b\beta-1}(1-\sigma^b)^{-3/2}\exp\{(Mh)(\sigma)\}\,\mathrm{d}\sigma}, \quad 0 < s < 1, \qquad (2.8)$$

where

$$K(h) = \frac{\beta\displaystyle\int_0^\infty |G(r\mathrm{e}^{-\mathrm{i}\pi/b})|\,\mathrm{d}r}{\pi\displaystyle\int_0^\infty |H(r\mathrm{e}^{-\mathrm{i}\pi/b})|\,\mathrm{d}r}, \qquad (2.9)$$

$$(Mh)(s) = \mathrm{Re}(\mathcal{M}h)(s) = b\int_0^1 \log\left|\frac{\tau^b - s^b}{\tau^b}\right| h(\tau)\,\mathrm{d}\tau. \qquad (2.10)$$

Observe that α does not appear in (2.8) because only $|G(s)|$ and $|H(s)|$ are used. Observe also that (since $(Mh)(0) = 0$)

$$\int_0^s \sigma^{b\beta-1}(1-\sigma^b)^{-3/2}\exp\{(Mh)(\sigma)\}\,\mathrm{d}\sigma \sim \frac{s^{b\beta}}{b\beta} \quad \text{as } s \downarrow 0,$$

so that the right-hand member of (2.8) tends to $K(h)$ as $s \downarrow 0$. Therefore $h(0) = K(h)$ for an admissible solution h of the integral equation.

In order to have a firm basis for the results in § 4 and § 5, we quote the following theorem, which is the main result of LP.

Theorem 2.1. *There exists a number β_1 in $(0, \frac{1}{4}]$ such that for $0 < \beta < \beta_1$ equation (2.8) has a solution $h = h(\cdot\,;\beta)$ with the following properties.*

(a) $K(h)$ is a finite positive number.

(b) h is admissible, so that $h(1) = 0$; also $(1-s)^{1/2}h'(s) < 0$ on $[0,1]$ and $h(0) = K(h)$.

(c) For each fixed s in $[0,1]$ the quantities $h(s;\beta)$, $(Mh)(s;\beta)$ and $K(h;\beta)$ depend continuously on β in $(0,\beta_1)$.

(d) The continuous function α defined by (2.3) satisfies $0 < \alpha(\beta) < \frac{1}{2}$ for $0 < \beta < \beta_1$ and $\alpha(\beta) \to \frac{1}{2}$ as $\beta \downarrow 0$, while $\alpha(\beta) \to 0$ as $\beta \uparrow \beta_1$.

(e) $h(0;\beta) \to \infty$ as $\beta \downarrow 0$.

We note that every wedge angle $2\pi\alpha$ in $(0,\pi)$ corresponds to some β in $(0,\beta_1)$, but unfortunately it is not proved in LP that α is a decreasing function of β.

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3. The limiting solution and a uniform approximation

Since h is a decreasing function of s (item (b) of the theorem) and $h(0;\beta) \to \infty$ as $\beta \downarrow 0$ (item (e)), we make the transformation

$$t = h(0;\beta)s, \quad 0 \leqslant s \leqslant 1, \quad g(t;\beta) = \frac{h(s;\beta)}{h(0;\beta)}, \tag{3.1}$$

in order to examine limits as $\beta \to 0$. (That β tends to zero from above has been emphasized enough.) If $g(t;\beta) \to g_0(t)$ as $\beta \to 0$, then formally (that is, without strict justification at this stage)

$$(Mh)(s_0) = b\int_0^{h(0;\beta)} \log\left|\frac{t^b - t_0^b}{t^b}\right| g(t;\beta)\,\mathrm{d}t$$
$$\to 2\int_0^{\infty} \log\left|\frac{t^2 - t_0^2}{t^2}\right| g_0(t)\,\mathrm{d}t \equiv m(t_0), \quad \text{say.}$$

Extending g_0 to the whole real line as an even function, and changing the first dummy variable in $\log|t+t_0| + \log|t-t_0| - 2\log|t|$, we obtain

$$m(t_0) = 2\int_{-\infty}^{\infty} \log\left|\frac{t - t_0}{t}\right| g_0(t)\,\mathrm{d}t \quad (-\infty < t_0 < \infty). \tag{3.2}$$

Now consider the integral equation (2.8). After an integration by parts it may be written

$$\frac{h(s)}{h(0)} = \frac{\exp\{-(Mh)(s)\}}{1 + \displaystyle\int_0^s [1 - (\sigma/s)^{b\beta}]\frac{\mathrm{d}}{\mathrm{d}\sigma}[(1-\sigma^b)^{-3/2}\exp\{(Mh)(\sigma)\}]\,\mathrm{d}\sigma} \tag{3.3}$$

for $0 < s < 1$. The formal limit of this, as $\beta \to 0$, is

$$g_0(t) = \exp\{-m(t)\}, \tag{3.4}$$

which combines with (3.2) to

$$m(t_0) = 2\int_{-\infty}^{\infty} \log\left|\frac{t - t_0}{t}\right| \mathrm{e}^{-m(t)}\,\mathrm{d}t \quad (-\infty < t_0 < \infty). \tag{3.5}$$

Equivalently,

$$m'(t_0) = -2\pi(T_{\mathrm{H}}\mathrm{e}^{-m})(t_0), \quad m(0) = 0,$$

where T_{H} denotes the Hilbert-transform operator (Titchmarsh 1948, ch. 5).

By means of the relationship between T_{H} and the Cauchy–Riemann equations, one finds that

$$m(t) = \log(\pi^2 t^2 + 1), \quad g_0(t) = \frac{1}{\pi^2 t^2 + 1} \quad (-\infty < t_0 < \infty), \tag{3.6}$$

and that this pair is the only solution of (3.5) and (3.4) under natural side conditions. The equation $h(0) = K(h)$ now shows that $h(0;\beta) \sim (2\pi^2\beta)^{-1/2}$ as $\beta \to 0$, and (2.3) shows that $\alpha \sim \frac{1}{2}$.

It is proved in LP that $g_0(t)$ is the genuine limit of solutions $g(t;\beta)$ as $\beta \to 0$ with $t \geqslant 0$ and t fixed (so that $s \to 0$). However, if $\beta \to 0$ with $s \geqslant$ const. > 0 (so that $t \to \infty$), then g_0 is not a good approximation. Returning to the integral equation (3.3), then to $h(0) = K(h)$ and to (2.3), one finds the improved approximations

$$h_1(s;\beta) = \frac{\sqrt{2\beta}}{\pi}\frac{(1-s^2)^{1/2}}{s^2+2\beta} \quad (0 \leqslant s \leqslant 1) \tag{3.7}$$

and

$$\alpha = \tfrac{1}{2} - \sqrt{2\beta} + O(\beta|\log\beta|). \tag{3.8}$$

Let $\delta = \sqrt{2\beta}$. We note that, after extension as an even function, h_1 resembles a delta function: $h_1(0;\beta) = (\pi\delta)^{-1}$, while $h_1(s;\beta) = O(\delta)$ for $s \geqslant$ const. > 0 and

$$\int_{-1}^{1} h_1(s;\beta)\,\mathrm{d}s = (1+\delta^2)^{1/2} - \delta.$$

Therefore the statement $h - h_1 = O(\delta(\log\delta)^2)$, while true, is not useful for $s \geqslant$ const. > 0. An error estimate that is both correct and informative cannot be simple; an example is

$$h(s;\beta) = h_1(s;\beta)\{1 + O(\delta^2(\log\delta)^2)\}, \quad 0 \leqslant s \leqslant 1-c, \tag{3.9 a}$$

$$h(s;\beta) = h_1(s;\beta) + O(\delta^3(\log\delta)^2), \quad 1-c \leqslant s \leqslant 1, \tag{3.9 b}$$

for any small positive number c independent of δ and s. In what follows, the phrase *to first order in* δ (where $\delta = \sqrt{2\beta}$) will imply an error factor $1 + O(\delta|\log\delta|)$, except possibly in small intervals of s near zeros of the function in question; *to second order in* δ will imply an error factor $1 + O(\delta^2(\log\delta)^2)$, with the same reservation.

4. The free boundary and flow along the wedge

(*a*) *Preliminaries*

The next step is to substitute the uniform approximation h_1 in (3.7) into the equations (2.5) for z_h and (2.6) for $u - \mathrm{i}v$. The integrations in those equations can be done explicitly if we approximate the integrands with an accuracy close to that of h_1. We shall use the notation

$$\delta = \sqrt{2\beta}, \quad \lambda = \frac{1}{2\delta}(1 - \tfrac{1}{2}\pi\delta), \tag{4.1}$$

recalling that $\alpha \sim \frac{1}{2} - \delta$. The significance of λ is that in the z-plane the wetted length $|\mathrm{AB}| \sim 2\lambda$ and that the largest pressures and velocity gradients occur near the midpoint

$$z_{\mathrm{E}} = \tfrac{1}{2}(z_{\mathrm{A}} + z_{\mathrm{B}}) \sim \lambda + \mathrm{i}(\tfrac{1}{2}\pi - 1)$$

of the wetted face AB. It is also in this neighbourhood of z_{E} that the free boundary BC comes closest to the vertex A and has very large curvature. All this has been predicted, long since, by heuristic theories (Cointe & Armand 1987; Howison *et al.* 1991) but it may be reassuring to see these results emerge from a different formulation and from strict analysis.

There is room for choice in the calculations. For example $s^{2\beta} \sim (s/\delta)^{2\beta}$ to second order in δ, since $\delta^{-2\beta} = 1 + O(\delta^2|\log\delta|)$. Again the function $h_{(1)}$ defined by

$$h_{(1)}(s;\beta) = \frac{\delta}{\pi}\left\{\frac{1}{s^2+\delta^2} - \frac{1-\sqrt{1-s^2}}{s^2} + \frac{\delta^2}{1+\delta^2}\right\}$$

is as good an approximation to h as h_1 is; although $h_{(1)}$ looks clumsier than h_1, it is integrated more easily. Thus, if we write $\gamma(s) = -\vartheta(a(s))$ for the angle of the free boundary BC (recall (1.13) and (2.2)), there are several routes to the result

$$\gamma(s) \sim 2\left\{\tan^{-1}\frac{1}{\delta} - \tan^{-1}\frac{s}{\delta} - \delta\left(\frac{1-\sqrt{1-s^2}}{s} - 1 + \tfrac{1}{2}\pi - \sin^{-1}s\right)\right\}, \tag{4.2}$$

and there are variants of this formula which are also correct to second order in δ.

We turn to the logarithmic potential of h_1. Using $b \sim 2$ and evaluating the derivative $(Mh_1)'(s)$ by contour integration for each of the following three cases, one finds that

$$(Mh_1)(s) \sim \begin{cases} \log\left(1+\dfrac{s^2}{\delta^2}\right) \quad \text{if } 0 \leqslant s \leqslant 1, \\ \log\dfrac{s^2}{\delta^2} - 2\delta\left(\cosh^{-1}s - \dfrac{\sqrt{s^2-1}}{s}\right) \quad \text{if } 1 \leqslant s < \infty, \\ 2\log\left(1+\dfrac{r}{\delta}\right) - 2\delta\left(\sinh^{-1}r - \dfrac{\sqrt{1+r^2}-1}{r}\right) \\ \qquad\qquad \text{if } s = r\mathrm{e}^{-\mathrm{i}\pi/b}, \quad 0 \leqslant r < \infty, \end{cases} \tag{4.3}$$

to second order in δ. It follows that, to the same order,

$$P \sim \tfrac{1}{2}\delta^2\lambda, \quad Q \sim \delta^{-1}\lambda, \quad z_{\mathrm{B}} = z_h(0) = 2\lambda + \mathrm{i}(\pi - 1) + O(\delta(\log\delta)^2), \tag{4.4}$$

where P and Q are the constants in (2.5)–(2.7).

(*b*) *The free boundary BC*

Here we approximate the integrand $G(s)$, with $0 \leqslant s < 1$, differently for $s \leqslant c\delta^{1/2}$ and for $s > c\delta^{1/2}$; in principle, the choice of the positive constant c (which is independent of s and δ) is immaterial because the two approximations to $z_h(s)$ differ only by $O(\delta)$ for $c_1\delta^{1/2} \leqslant s \leqslant c_2\delta^{1/2}$, and $O(\delta)$ is the error of both, apart from logarithms.

Omitting the label h from z_h, and with the understanding that s is real and in $[0, 1)$, we obtain

for $0 \leqslant s \leqslant c\delta^{1/2}$,
$$x(s) - x_{\mathrm{B}} \sim -\lambda s^{2\beta}(1 - \tfrac{1}{2}s^2), \tag{4.5 a}$$
$$y(s) - y_{\mathrm{B}} \sim -2\delta\lambda s^{2\beta}(\tfrac{1}{2}\pi + s); \tag{4.5 b}$$

for $c\delta^{1/2} \leqslant s < 1$,
$$x(s) - x(c\delta^{1/2}) \sim \lambda\{(1-s^2)^{-1/2} - (1-c^2\delta)^{-1/2}\}, \tag{4.5 c}$$
$$y(s) - y(c\delta^{1/2}) \sim 2\delta\lambda\left\{\frac{\frac{1}{2}\pi - \sin^{-1}s}{\sqrt{1-s^2}} - \frac{\frac{1}{2}\pi - \sin^{-1}(c\delta^{1/2})}{\sqrt{1-c^2\delta}}\right\}. \tag{4.5 d}$$

Figure 2 shows the free boundary for $\delta = 0.05$, that is, for a wedge semi-angle $\pi\alpha$ of approximately 81°; then $\lambda = 9.215$.

(*c*) *Flow along the wedge face AB*

We rotate the coordinate frame, writing

$$\xi + \mathrm{i}\eta = (x + \mathrm{i}y)\mathrm{e}^{-\mathrm{i}(\pi/2-\pi\alpha)}, \quad W_1 + \mathrm{i}W_2 = (u + \mathrm{i}v)\mathrm{e}^{-\mathrm{i}(\pi/2-\pi\alpha)}, \tag{4.6}$$

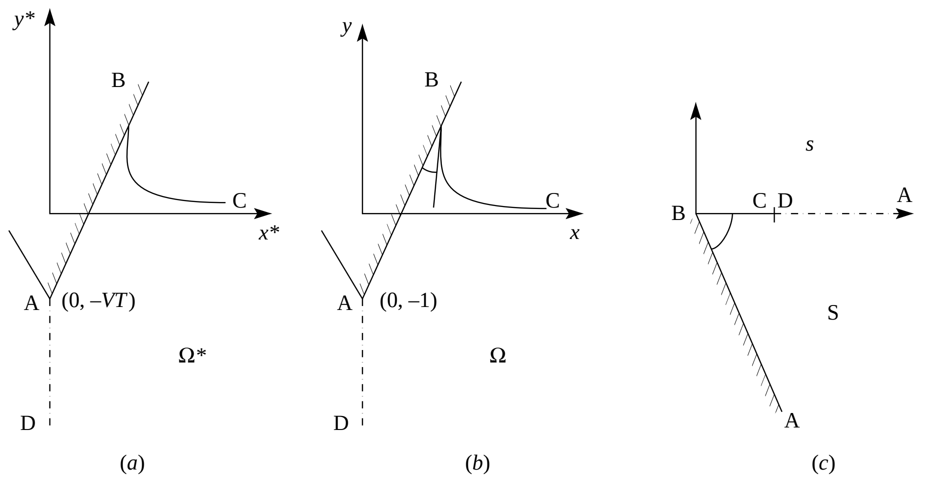

Figure 2. The free boundary for $\delta = 0.05$.

so that grad ξ and W_1 are parallel to the wedge face AB. (To our accuracy, ξ and x cannot be distinguished on AB, but it is helpful to begin with exact equations.) Note that $(W_1+\mathrm{i}W_2)_\mathrm{A} = (\xi+\mathrm{i}\eta)_\mathrm{A}$, because (1.2) and (1.3) imply that $(u+\mathrm{i}v)_\mathrm{A} = -\mathrm{i} = z_\mathrm{A}$, and that $(W_1+\mathrm{i}W_2)_\mathrm{B} = (\xi+\mathrm{i}\eta)_\mathrm{B}$, because (1.9) with $a = 0$ implies that $z_\mathrm{B} = (u+\mathrm{i}v)_\mathrm{B}$.

Restricting attention henceforth to AB, on which $s = r\mathrm{e}^{-\mathrm{i}\pi/b}$, $0 \leqslant r < \infty$, we shall proceed from the equations

$$\frac{\mathrm{d}\xi}{\mathrm{d}W_1} = -\frac{\mathrm{d}|z(r\mathrm{e}^{-\mathrm{i}\pi/b}) - z_\mathrm{B}|}{\mathrm{d}r} \bigg/ \frac{\mathrm{d}W_1}{\mathrm{d}r}$$

$$\sim \tfrac{1}{2}\delta^3 \frac{(1+r/\delta)^4}{r(1+r^2)^{3/2}} \left\{1 - 4\delta\left(\sinh^{-1} r - \frac{\sqrt{1+r^2}-1}{r}\right)\right\}, \tag{4.7}$$

$$\frac{\mathrm{d}W_1}{\mathrm{d}r} \sim -\frac{2\lambda}{\delta} r^{2\beta}(1+r/\delta)^{-2}\left\{1 + 2\delta\left(\sinh^{-1} r - \frac{\sqrt{1+r^2}-1}{r}\right)\right\}. \tag{4.8}$$

For $\delta \log r \to \infty$, the last factor should be replaced by

$$(2r)^{2\delta}\left\{1 + 2\delta\left(\sinh^{-1} r - \log 2r - \frac{\sqrt{1+r^2}-1}{r}\right)\right\},$$

and similarly in (4.7). We stress that r *denotes* $|s|$ (on AB) *and not* $|z|$.

Figure 3 shows implications of (4.7) and (4.8), again for $\delta = 0.05$. It must be emphasized, however, that *figure 3 has acceptable accuracy near the midpoint* $\xi_\mathrm{E} \sim \lambda$ *only as a graph of* $\xi(W_1)$; the graphs of $W_1(\xi)$ and $p(\xi)$ in figure 3 may contain large errors and are drawn only to give a qualitative impression.

Nevertheless, it is possible to calculate the force on the wedge to second order in δ. Our method is

(a) to regard both ξ and the pressure (on AB) as functions of W_1, which is legitimate because (2.5) and (2.6) show that $\mathrm{d}\xi/\mathrm{d}W_1 > 0$ on AB;

(b) to use two distinct formulae for the pressure, one having A as reference point and the other using B.

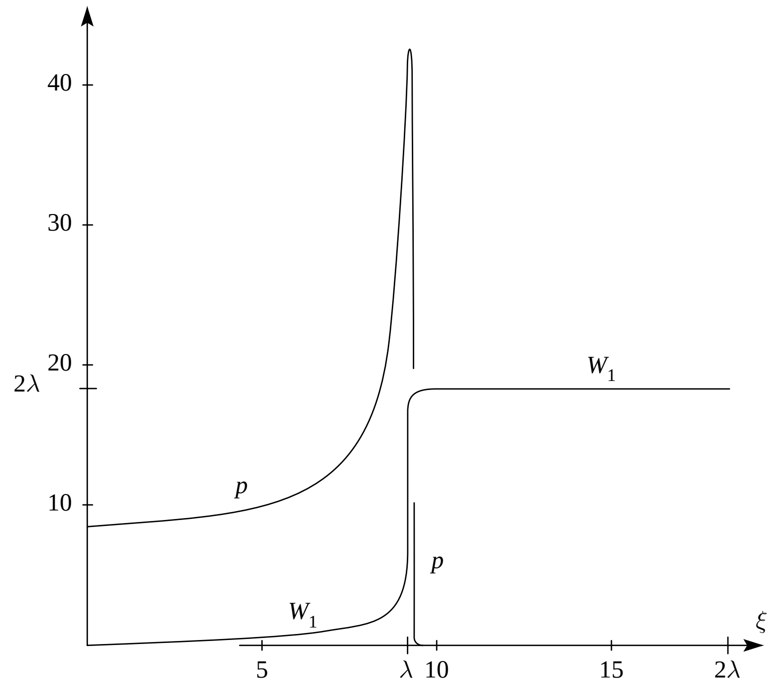

Figure 3. Tangential velocity and pressure on the wedge face for $\delta = 0.05$.

To this end, define

$$2L = |\mathrm{AB}| - \cos\pi\alpha = \xi_{\mathrm{B}} = W_{1,\mathrm{B}} \qquad (L \sim \lambda \text{ as } \delta \to 0),$$

$$w = \frac{W_1}{2L} \quad \text{on AB} \qquad \left(w_{\mathrm{A}} \leqslant w \leqslant 1,\ w_{\mathrm{A}} = -\frac{\cos\pi\alpha}{2L} \sim -\pi\delta^2\right),$$

$$X(w) = X(w;\delta) = \xi/L \quad \text{on AB} \qquad (2w_{\mathrm{A}} \leqslant X(w) \leqslant 2).$$

Adding the definition $p^*(x^*, y^*, T) = \rho V^2 p(x, y)$ of the reduced pressure p to the similarity transformation (1.8), and normalizing by $p_{\mathrm{BC}} = 0$, we find from the ξ-component of the reduced momentum equation (and from $\partial W_1/\partial\eta = \partial W_2/\partial\xi = 0$ on AB) that

$$\frac{\partial}{\partial\xi}(p + \tfrac{1}{2}W_1^2) = \xi\frac{\partial W_1}{\partial\xi} \quad \text{on AB}.$$

Hence, for $w_{\mathrm{A}} \leqslant w_0 \leqslant 1$,

$$p_{\mathrm{AB}}(w_0) = p_{\mathrm{A}} + 2L^2\left\{w_{\mathrm{A}}^2 - w_0^2 + \int_{w_{\mathrm{A}}}^{w_0} X(w)\,\mathrm{d}w\right\}, \tag{4.9}$$

and

$$p_{\mathrm{AB}}(w_0) = 2L^2\left\{1 - w_0^2 - \int_{w_0}^{1} X(w)\,\mathrm{d}w\right\}. \tag{4.10}$$

To calculate p_{A} one notes that $p(x, y) \to 0$ as $|z| \to \infty$ because $p_{\mathrm{BC}} = 0$ and $|\nabla p(x, y)|$ is $O(|z|^{-2})$; then an integration along DA yields

$$\hat{p}_{\mathrm{A}} \equiv p_{\mathrm{A}}/L^2 = 2\delta + O(\delta^3|\log\delta|). \tag{4.11}$$

The upward force on the whole wedge is

$$F^* = 2\rho V^3 T \sin\pi\alpha \int_{\xi_A}^{\xi_B} p\,\mathrm{d}\xi = 2\rho V^3 T \sin\pi\alpha\, L^3 \mathcal{F} \tag{4.12}$$

if we define

$$\mathcal{F} = \int_{w_A}^{1} \hat{p}(w) X'(w)\,\mathrm{d}w, \quad \hat{p}(w) = p_{AB}(w)/L^2. \tag{4.13}$$

Recall that $L \sim \lambda$ to second order in δ.

The task of finding $X(w)$ to second order is helped by scrutiny of (4.7) and (4.8) in three particular intervals: (i) $r \geqslant c_1 > 0$; (ii) $c_2\delta \leqslant r \leqslant C_2\delta$; (iii) $0 \leqslant r^{2\beta} \leqslant 1 - c_3$ $(0 < c_3 < 1)$. Here c_1, c_2, C_2 and c_3 are arbitrary positive numbers (subject to $c_2 < C_2$ and $c_3 < 1$) but *independent of* δ *and* r. Notice that, if $r = \delta^m$ with $0 < m < 1$, then r is strictly between (i) and (ii) as $\delta \to 0$, hence not in any of (i)–(iii); if $r = \delta^n$ with $1 < n < \infty$, then r is strictly between (ii) and (iii), hence not in any of (i)–(iii). The images in the z-plane of the intervals (i), (ii) and (iii) correspond, respectively, to the parts of AB in the regions called I, II and III by Howison *et al.* (1991). With E denoting the midpoint of AB in the z-plane, we shall find that, for us, region I is a part of AE bounded away from E, region II is a neighbourhood of E that has X-length of order δ^2, and region III is a part of EB bounded away from E.

Once (4.7) and (4.8) are understood in the intervals (i), (ii) and (iii), one can construct an overall or composite approximation to $X'(w)$ partly by known methods and partly by improvisation. Integration of this derivative, with use of either $X_A = 2w_A$ or $X_B = 2$, yields an approximation to $X(w)$.

To first order in δ, all this is straightforward. The value $w_A \sim -\pi\delta^2$ is replaced by zero and one finds that

$$X(w;\delta) \sim \frac{w}{\sqrt{w^2+\delta^2}} + 1 - (1-w)^{2\beta} \quad (0 \leqslant w \leqslant 1), \tag{4.14}$$

$$\hat{p}(w;\delta) \sim 2(\sqrt{w^2+\delta^2} - w^2), \quad \mathcal{F} = \pi\delta + O(\delta^2|\log\delta|). \tag{4.15}$$

In region I, $w \leqslant \text{const.}\,\delta$ and $X \leqslant \text{const.} < 1$; hence

$$X \sim \frac{w}{\sqrt{w^2+\delta^2}}, \quad \hat{p} \sim 2\sqrt{w^2+\delta^2} \sim \frac{2\delta}{\sqrt{1-X^2}} \quad \text{in region I.}$$

Since $2\delta\int_0^1 (1-X^2)^{-1/2}\,\mathrm{d}X = \pi\delta$, the force $\mathcal{F}$ can be said to come from region I, to the first order, even though the point $X = 1$ is in region II.

5. Pressure and force to second order

For the wedge face AB, the basic regions may now be described as

I : $w_A \leqslant w \leqslant \text{const.}\,\delta$;

II : $0 < \text{const.} \leqslant w \leqslant \text{const.} < 1$;

III : $0 \leqslant (1-w)^{2\beta} \leqslant \text{const.} < 1$,
so that $1 - w \leqslant \exp\{(-1/2\beta)\log(1/k)\}$ for some $k < 1$.

This is consistent with figure 3. The importance of region I prompts the definition $q = (w - w_A)/\delta$ (where $w_A \sim -\pi\delta^2$).

Then, for $w_{\mathrm{A}} \leqslant w \leqslant 1$ or $0 \leqslant q \leqslant \delta^{-1}(1-w_{\mathrm{A}})$,

$$X(w;\delta) = \frac{q}{\sqrt{q^2+1}} - \delta\left\{\frac{2q^3+4q}{(q^2+1)^{3/2}}\sinh^{-1}\frac{1}{q} + \frac{1}{(q^2+1)^{3/2}}\sinh^{-1} q - \frac{q}{q^2+1}\right\}$$
$$+\delta^2\left(\sinh^{-1} q - \frac{q}{\sqrt{q^2+1}}\right) + 1 - (1-w)^{2\beta} + O(\delta^2(\log\delta)^2). \quad (5.1)$$

The δ^2 term is not significant; it is retained because its derivative does play a part in the formula for $\mathrm{d}X/\mathrm{d}w$, from which (5.1) has been derived and which is used for calculation of $\mathcal{F}$. The term $1-(1-w)^{2\beta}$ is significant only in region III and in a part of the region between II and III.

From (4.9), (4.11) and (5.1) we find that, for $w_{\mathrm{A}} \leqslant w \leqslant 1$ or $0 \leqslant q \leqslant \delta^{-1}(1-w_{\mathrm{A}})$,

$$\hat{p}(w;\delta) = 2\delta\left\{\sqrt{q^2+1} - \delta q^2 - \delta\left(\frac{2q^2}{\sqrt{q^2+1}}\sinh^{-1}\frac{1}{q} + \frac{q}{\sqrt{q^2+1}}\sinh^{-1} q\right)\right\}$$
$$+O(\delta^3(1+q)(\log\delta)^2). \quad (5.2)$$

Evidently $\hat{p}$ grows from order δ in region I ($q \leqslant$ const.) to order one in region II, apart from factors $\log\delta$, which we ignore in this discussion. (Of course not all terms in (5.2) grow at the same rate as q increases to order δ^{-1}.) Correspondingly, the error term in (5.2) grows from order δ^3 in region I to order δ^2 in regions II and III. If the approximation (5.2) is used up to $w=1$, then $\mathcal{F}$ is *not* found correctly to second order, because of an error of order δ^2 over an X-length of approximately 1 (the length of the regions beyond II). This difficulty is resolved by the inequality

$$|\hat{p}(w)| \leqslant \text{const.}(1-w) \leqslant \text{const.}\,\delta^m \quad \text{if } 1-\delta^m \leqslant w \leqslant 1, \quad (5.3)$$

in which $\hat{p}$ denotes the exact pressure function and not the explicit part of (5.2). This inequality is an immediate consequence of (4.10), since $X(w)$ is bounded by $w_{\mathrm{A}} \leqslant X(w) \leqslant 2$. Then (4.13) shows, since $X'(w) > 0$ and $X(w)$ is bounded, that

$$\mathcal{F} = \int_{w_{\mathrm{A}}}^{1-\delta^4} \hat{p}(w)X'(w)\,\mathrm{d}w + O(\delta^4). \quad (5.4)$$

We have chosen $m=4$; any value $m \geqslant 3$ would serve. The significance of (5.4) is that this small reduction of the w-interval almost halves the length of the corresponding X-interval, and excises region III.

The approximation (5.2) and (5.4) yield

$$\mathcal{F} = \pi\delta - 4\delta^2(2G-1) + O(\delta^3(\log\delta)^2), \quad (5.5)$$

where

$$G = 1 - \frac{1}{3^2} + \frac{1}{5^2} - \frac{1}{7^2} + \cdots \approx 0.916 \quad \text{(Catalan's constant).}$$

The integration leading to (5.5) is elaborate, but has been done in two quite different ways, as have the derivations of (5.1) and (5.2).

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